Eur. Phys. J. A **25**, 413–418 (2005)
DOI 10.1140/epja/i2005-10129-8

THE EUROPEAN PHYSICAL JOURNAL A

A semiclassical formula for the reaction cross-section of heavy ions

F. Cerutti[1,2], A. Clivio[3], and E. Gadioli[1,2,a]

[1] Dipartimento di Fisica, Università di Milano, Milano, Italy
[2] INFN, Sezione di Milano, Milano, Italy
[3] Scuola di Specializzazione in Fisica Sanitaria, Università di Milano, Milano, Italy

Received: 11 March 2005 / Revised version: 4 July 2005 /
Published online: 21 September 2005 –
Communicated by A. Molinari

Abstract. We discuss an improvement to the expression for the heavy-ion reaction cross-section proposed long ago by Karol (P.J. Karol, Phys. Rev. C **11**, 1203 (1975)) which describes the two-ion interaction through the nucleon-nucleon collisions in their overlap region when they come into contact. This improvement consists in considering the limitations due to the Pauli principle, which at low incident energies forbids many nucleon interactions, and in including a Coulomb factor. The final expression for which we also propose a new parametrization reproduces satisfactorily the known reaction cross-sections for many ion combinations using a global set of parameters. A decomposition of the reaction cross-section into the contributions of peripheral interactions, as a function of the impact parameter, is also discussed.

PACS. 25.70.-z Low and intermediate energy heavy-ion reactions

1 Introduction

The reaction cross-section, σ_R, is a basic information in nuclear dynamics. However, in heavy-ion interactions, for many interaction partners, σ_R is not experimentally known and one must resort to theoretical estimations. These could be done with the optical model which, however, for describing the real and imaginary potentials requires sets of parameters for which, in most cases, raccomended prescriptions do not exist. For this reason one often resorts to suitably parametrized semiempirical expressions of σ_R such as those, for instance, based on the Bradt-Peters or Renberg forms [1–4] or the Fresnel model [5–7]. Other estimates of σ_R are based on more elaborated calculations. Among these we deem especially interesting those based on the hypothesis that the interaction of two nuclei may be described by means of the interactions of their constituent nucleons (see, for instance, [8–19]). In particular, the result of Karol [8] seems attractive to us (which has been further elaborated by other authors [14–17,19]); he was able to produce a simple analytical expression of σ_R using a Gaussian density distribution for the interacting ions. Another interesting approach was worked out by the authors of [18] using a different approximation for the nuclear density distributions. In this paper we wish to discuss some ameliorations to the Karol's expression which are effective in allowing a considerably better reproduction, than that afforded by the original formula, of the experimental reaction cross-sections at low energies.

[a] e-mail: gadioli@mi.infn.it

2 A semiclassical calculation of heavy-ion reaction cross-sections

The Karol model [8] is a generalization of the semiclassical optical model of Fernbach, Serber and Taylor [20] and expresses σ_R by means of the transparency function $T(b)$, *i.e.* the probability that at an impact parameter b the projectile goes through the target without interacting:

$$\sigma_R = 2\pi \int_0^\infty (1 - T(b))\, b\, \mathrm{d}b\,. \tag{1}$$

$T(b)$ is evaluated considering the nucleon-nucleon collisions in the overlap region of the projectile and the target.

The calculation of $T(b)$ and the main assumptions of the model are described in the original paper [8]. Here we simply remind what is needed to present our modifications.

Considering a coordinate system with origin in the center of the target nucleus and the z-axis along the beam direction, for $T(b)$ one has

$$T(b) = \exp\left(-\int_{-\infty}^{\infty} Q(b,z)\,\mathrm{d}z\right)\,, \tag{2}$$

where the thickness function $Q(b,z)$ gives the interaction probability per unit path length [21].

In order to get an analytical solution for σ_R, the target and projectile density distributions involved in the expression of $Q(b,z)$ are assumed to be Gaussian:

$$\rho(r) = \rho(0)\, \exp\left(-(r/a)^2\right) . \qquad (3)$$

In the original paper a distinction was made between light ($A \le 40$) and heavy ($A > 40$) nuclei. We do not retain such assumption and for all nuclei we choose the parameters $\rho(0)$ and a to reproduce in the surface region the Fermi nuclear density distribution

$$\rho(r) = \rho_0 \left[1 + \exp\left((r-c)/d\right)\right]^{-1} , \qquad (4)$$

where, with reference to the original notation of Karol [8],

$$d = \frac{t}{4.4} . \qquad (5)$$

The parameters t and ρ_0 are kept independent of the particular considered nucleus and their values are taken equal to 2.4 fm and 0.17 nucleons/fm^3, respectively, while the half-central-density radius c is evaluated for each nucleus by imposing that

$$\int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{\rho_0}{1+e^{(r-c)/d}}\, r^2 \sin\theta\, \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi = A . \qquad (6)$$

The found c values range from 0.954 $A^{1/3}$ fm for ^{12}C to 1.095 $A^{1/3}$ fm for ^{208}Pb. The mass dependence of $r_0 = c/A^{1/3}$ is shown in fig. 1.

The Gaussian distributions (3) are not normalized (*i.e.* their integration does not provide the correct number of nucleons of the considered ion). However, as noted by Karol [8], they give the correct number of nucleons in the surface region of the nuclei, which —in practice— is the only part of the density distribution which affects the numerical calculation of the reaction cross-section. The use of Gaussian density distributions provides the following analytical expression for $T(b)$:

$$-\ln\left(T(b)\right) = \frac{\pi^2 \overline{\sigma}(E_{\rm nucl})\rho_T(0)\rho_P(0)a_T^3 a_P^3}{a_T^2 + a_P^2} \times \exp\left(-b^2/(a_T^2+a_P^2)\right) , \qquad (7)$$

where the subscripts T and P refer to the target and the projectile, respectively, a and $\rho(0)$ are the Gaussian distribution parameters, the expressions of which in terms of ρ_0, c and t are given in table 1, and $\overline{\sigma}$ is the average nucleon-nucleon cross-section:

$$\overline{\sigma}(E_{\rm nucl}) = \left[\left(\frac{Z_T}{A_T}\right)\left(\frac{Z_P}{A_P}\right) + \left(\frac{N_T}{A_T}\right)\left(\frac{N_P}{A_P}\right)\right] \sigma^{\rm free}_{pp\,(nn)}(E_{\rm nucl}) + \left[\left(\frac{Z_T}{A_T}\right)\left(\frac{N_P}{A_P}\right) + \left(\frac{Z_P}{A_P}\right)\left(\frac{N_T}{A_T}\right)\right] \sigma^{\rm free}_{np}(E_{\rm nucl}) , \qquad (8)$$

where $E_{\rm nucl} = E_{\rm Lab}/A_P$, being $E_{\rm Lab}$ the projectile energy in the laboratory, and $\sigma^{\rm free}(E_{\rm nucl})$ are the free nucleon-nucleon cross-sections. This essentially means that one

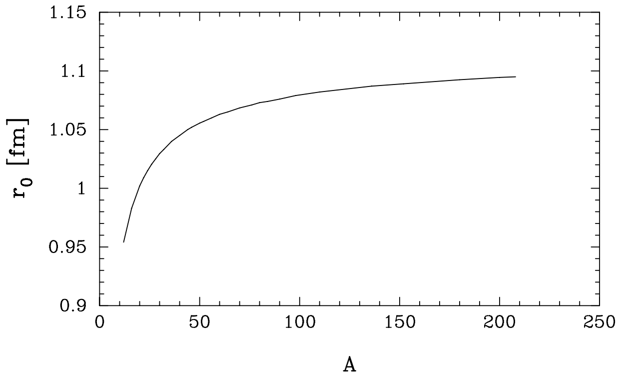

Fig. 1. Variation of $r_0 = c/A^{1/3}$ with the ion's mass number.

Table 1. Parameters of the Gaussian nuclear density distributions.

a	$\rho(0)$
$\left((4ct+t^2)/k\right)^{1/2}$	$\frac{1}{2}\rho_0 \exp(c/a)^2$
c is obtained from (6) $t = 2.4$ fm $k = 4\ln 5 = 6.43775\ldots$	$\rho_0 = 0.17$ nucleons/fm^3

assumes that the projectile nucleons, with energy $E_{\rm nucl}$, interact independently from each other with the target nucleons. For the free nucleon-nucleon cross-sections we use the expressions given in [14] (at $E_{\rm nucl} > 40$ MeV) and [22] (for $\sigma^{\rm free}_{np}$ at $E_{\rm nucl} \le 40$ MeV). We take $\sigma^{\rm free}_{pp\,(nn)} = \sigma^{\rm free}_{np}/3$ at $E_{\rm nucl} < 20$ MeV.

The use of (7) in (1) leads to the following expression for the reaction cross-section:

$$\sigma_R = \pi(a_T^2 + a_P^2)\left(E_1(\chi) + \ln\chi + \gamma\right) , \qquad (9)$$

where $\gamma = 0.5772\ldots$ is Euler's constant,

$$\chi = \frac{\pi^2 \overline{\sigma}(E_{\rm nucl})\rho_T(0)\rho_P(0)a_T^3 a_P^3}{a_T^2 + a_P^2} , \qquad (10)$$

and $E_1(\chi)$ is the exponential integral [23]

$$E_1(\chi) = \int_\chi^\infty \frac{e^{-u}}{u}\, \mathrm{d}u . \qquad (11)$$

Karol showed that expression (9), with his original parametrization, allows one to satisfactorily reproduce σ_R for many heavy-ion reactions at incident energies of a few GeV/nucleon [8]. However, it works much less satisfactorily at low energies where it overestimates sizeably the known experimental values of σ_R. This shortcoming was noted by many authors who worked out various ameliorations of the original Karol's formula.

Charagi and Gupta [14] modified the formula by introducing a Gaussian finite-range interaction and above all a Coulomb factor reducing the reaction cross-section at low energies. They however did not introduce a physically important effect such as the limitations imposed by the Pauli principle (PP) which at low energy forbids many nucleon-nucleon interactions.

A way to consider the medium effect was proposed in [17] using a phenomenological formula for the in-medium nucleon-nucleon cross-section. A similar approach has been also recently adopted in [19], where a linear density-dependent form of the nucleon-nucleon cross-section is used which reduces the free cross-section by a quantity proportional —through an empirical density parameter β— to the interacting ion nuclear densities.

We propose in this paper to take into account the in-medium effects on the nucleon-nucleon cross-section by multiplying in (8) the free nucleon-nucleon cross-section by a Pauli factor [24,25] which depends on the incident ion energy and the Fermi energy in the two-ion overlap region.

In a local density approximation, we might define a local value of the Fermi energy depending on the impact parameter and the density distributions of the two interacting nuclei. However, introducing in a consistent way the PP correction would lead to a considerable complication destroying the intrinsic simplicity of the model. Thus, we attempt an alternative way by introducing in the expression of $\overline{\sigma}$ a Pauli factor depending on an average Fermi energy $\overline{E_{\rm F}}$ varying with the incident ion energy.

If the two nuclei could be described as two Fermi gases with reduced radius $r_{0\rm F}$, the average Fermi energy would be

$$\overline{E_{\rm F}} = \frac{48.5\ \text{MeV}\,\text{fm}^2}{r_{0\rm F}^2} \approx 28.7\,\text{MeV}\,, \tag{12}$$

using $r_{0\rm F} \approx 1.3$ fm. In fact, nuclei have not the sharp Fermi gas density distributions and, considering that with increasing energy the contribution of peripheral collisions to σ_R becomes progressively dominant, we have to expect that with increasing energy the values of $\overline{E_{\rm F}}$ to be used in the local density approximation may be substantially smaller being proportional to $[\rho(r)]^{2/3}$. The analysis of experimental σ_R behavior for a significant number of systems suggests for the $\overline{E_{\rm F}}$ dependence on bombarding energy the expression

$$\overline{E_{\rm F}}(E_{\rm nucl}) = B\exp(-KE_{\rm nucl}) + C \tag{13}$$

with $B = 20.7$ MeV, $K = 0.06$ MeV^{-1}, and $C = 8.0$ MeV.

On this basis we multiply the average nucleon-nucleon cross-section (8) by the Pauli factor

$$\begin{aligned} P(\xi) &= 1 - \tfrac{7}{5}\xi && \text{if } \xi \le \tfrac{1}{2}\,, \\ P(\xi) &= 1 - \tfrac{7}{5}\xi + \tfrac{2}{5}\xi(2 - \tfrac{1}{\xi})^{5/2} && \text{if } \xi \ge \tfrac{1}{2} \end{aligned} \tag{14}$$

with

$$\xi = \frac{\overline{E_{\rm F}}}{E_{\rm nucl} + V}\,. \tag{15}$$

The local potential depth V is given by [24,25]

$$\begin{aligned} V = \overline{E_{\rm F}} + \Big(&S_p(Z_T,N_T) + S_n(Z_T,N_T) \\ &+ S_p(Z_P,N_P) + S_n(Z_P,N_P)\Big)/4\,, \end{aligned} \tag{16}$$

where $S_{p\,(n)}$ is the proton (neutron) separation energy.

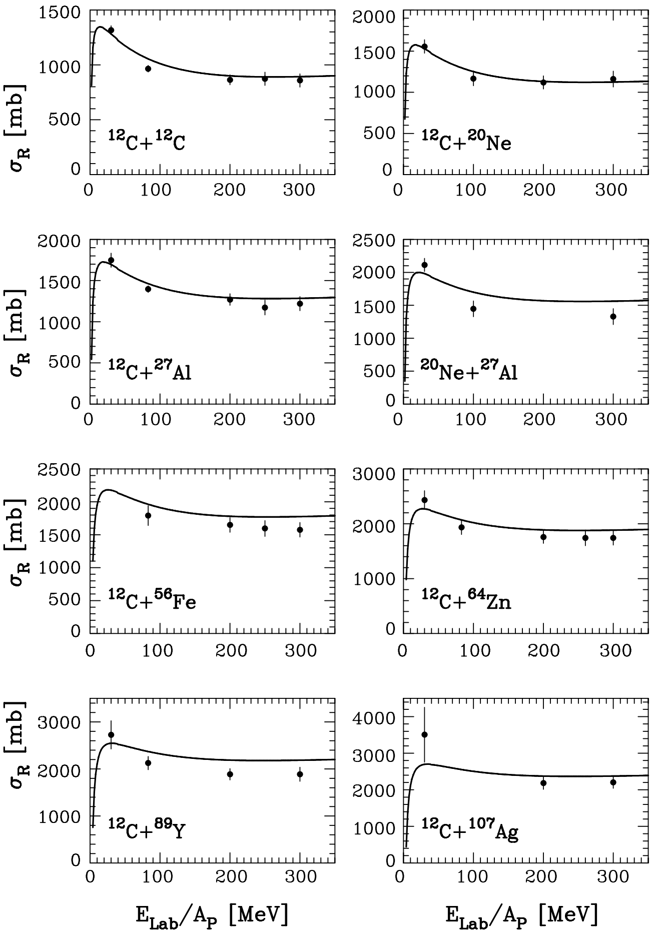

Fig. 2. Reaction cross-section for different heavy-ion pairs. The experimental values [26–32] are given by the full circles, the theoretical estimate by the full line.

As noted before [14], the reaction cross-section expression (9) does not contain any Coulomb term taking into account that at energies approaching the Coulomb barrier the two-ion nuclear interaction is reduced or even forbidden. In order to preserve the simplicity of the formula, we simply introduce the classical factor for the Coulomb term giving

$$\sigma_R = \pi(a_T^2 + a_P^2)\,(E_1(\chi) + \ln\chi + \gamma)\,(1 - V_C/E_{\rm ch}) \tag{17}$$

where

$$E_{\rm ch} = E_{\rm nucl}\frac{A_P A_T}{A_P + A_T} \tag{18}$$

and, using the expression suggested by [6],

$$V_C = \frac{Z_P Z_T\, e^2}{R_P + R_T + D} - \frac{Q\, R_P R_T}{R_P + R_T} \tag{19}$$

with $D = 3.2$ fm,

$$R_i = 1.12\,A_i^{1/3} - 0.94\,A_i^{-1/3}\ \text{fm} \qquad i = P, T\,, \tag{20}$$

and $Q = 1.0$ MeV fm^{-1}.

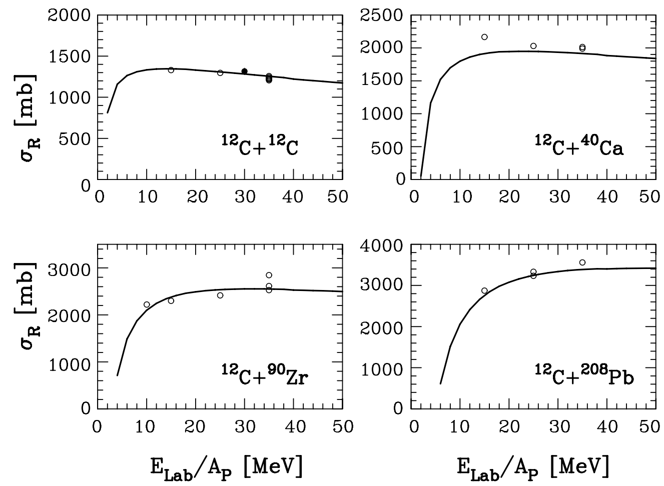

Fig. 3. Comparison between our reaction cross-section estimates (full line) and the predictions by Optical Model calculations (empty circles) using parameters from a best fitting of elastic scattering differential cross-sections [33]. Where we report two or three values at the same energy for the Optical Model predictions, they correspond to different sets of parameters giving an equivalent reproduction of elastic scattering data. The full circle in the top left frame, at $E_{\rm Lab}/A_P = 30$ MeV, represents the experimental value of [31].

Figure 2 shows a comparison of experimental [26–32] and calculated reaction cross-sections for reactions induced by light projectiles. The agreement is quite satisfactory especially considering the simplicity of the model and the use of parameters which are not fitted to the particular considered reaction.

At low energies our prediction agrees very satisfactorily with the σ_R values calculated with the Optical Model using parameters obtained by a best fitting of the elastic scattering angular distributions [33], as shown in fig. 3.

3 Contributions of peripheral interactions to reaction cross-section

It seems quite natural to use the model for evaluating —at not too low incident energies— i) the contribution to σ_R of peripheral reactions involving only a limited overlap of projectile and target and ii) the number of projectile and target nucleons in the overlap region.

The evaluation of the partial reaction cross-section for the impact parameter interval (b', b'') is straightforward:

$$\sigma_R(b',b'') = \pi(a_T^2 + a_P^2)\Big(E_1(\chi') - E_1(\chi'') + \ln\chi' - \ln\chi''\Big)\,(1 - V_C/E_{\rm ch})\;, \qquad (21)$$

where

$$\chi' = \chi \exp\left(-(b')^2/(a_T^2 + a_P^2)\right) \qquad (22)$$

and

$$\chi'' = \chi \exp\left(-(b'')^2/(a_T^2 + a_P^2)\right)\;. \qquad (23)$$

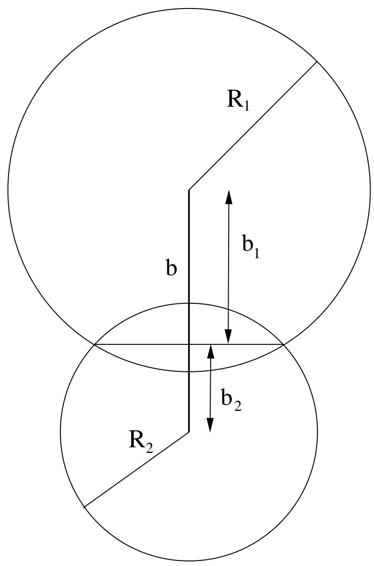

Fig. 4. Schematization of a peripheral heavy-ion interaction.

Slightly more involved is the calculation of the number of the projectile and target nucleons in the overlap region as a function of the impact parameter. To do that we treat the nuclei as spheres of radius R defined through the relation

$$\int_0^R\int_0^\pi\int_0^{2\pi} \frac{\rho_0}{1+e^{(r-c)/d}}\, r^2 \sin\theta\, {\rm d}r{\rm d}\theta{\rm d}\phi = A - \delta \qquad (24)$$

using for δ the value $A/100$. With reference to fig. 4, the overlap region between projectile and target is the union of two spherical segments, one belonging to the sphere of radius R_1 (projectile) and the other to the sphere of radius R_2 (target). Their heights are $h_1 = R_1 - b_1$ and $h_2 = R_2 - b_2$, where

$$b_1 = \frac{b^2 + R_1^2 - R_2^2}{2b} \qquad (25)$$

and

$$b_2 = \frac{b^2 - R_1^2 + R_2^2}{2b}\;, \qquad (26)$$

being b the impact parameter.

The number of interacting nucleons of projectile is given by $A_{P,1} + A_{P,2}$ where

$$A_{P,1} = \int_{b_1}^{R_1}\int_0^{\arccos(b_1/r)}\int_0^{2\pi} \frac{\rho_0}{1+\exp\left((r - c_P)/d\right)} \times r^2\sin\theta\,{\rm d}r{\rm d}\theta{\rm d}\phi \qquad (27)$$

and

$$A_{P,2} = \int_{b_2}^{R_2}\int_0^{\arccos(b_2/r)}\int_0^{2\pi} \frac{\rho_0}{1+\exp\left((\sqrt{r^2+b^2-2br\cos\theta} - c_P)/d\right)}\, r^2\sin\theta\,{\rm d}r{\rm d}\theta{\rm d}\phi\,. \qquad (28)$$

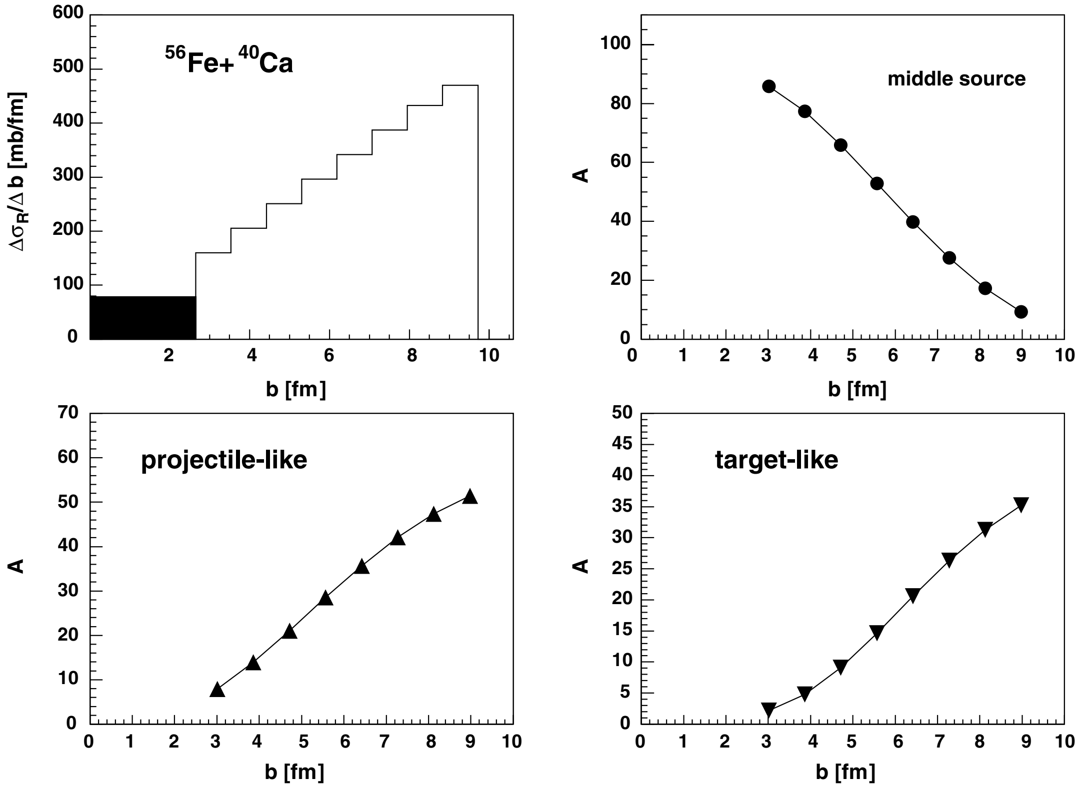

Fig. 5. Decomposition of the reaction cross-section for the interaction of a 25 MeV/nucleon ^{56}Fe beam with ^{40}Ca into the contributions corresponding to subsequent impact parameter intervals (top left) and estimates, as a function of the impact parameter, of the total number of nucleons in the overlap region (top right) and the number of the projectile's (bottom left) and target's (bottom right) nucleons which act as spectators.

Substituting the subscripts P and 1 for T and 2, respectively, we obtain from (27) the number of target nucleons in the spherical segment of height h_2 and from (28) the number of those in the spherical segment of height h_1.

An example of the results which one gets is illustrated in fig. 5 for the interaction of a 25 MeV/nucleon ^{56}Fe beam with ^{40}Ca. The top left frame shows the decomposition of σ_R into the contributions of subsequent impact parameter intervals. The black area is the expected contribution of complete fusion reactions, tentatively evaluated using the critical angular momentum predicted by the rotating liquid drop model [34]. The top right frame shows the total number of nucleons in the overlap region as a function of the impact parameter. The bottom left (right) frame shows, as a function of the impact parameter, the number of the projectile (target) nucleons which act as spectators and constitute the projectile-like (target-like) fragment. These results are representative of the fact that already at $E_{\mathrm{nucl}} = 25$ MeV most of the reactions occur at large impact parameters and involve a number of nucleons significantly smaller than the total nucleon number.

4 Conclusions

The inclusion of an *average* Pauli blocking factor and the classical Coulomb factor into the analytical expression proposed by Karol [8], with the use of the parametrization proposed above, allows one to reproduce satisfactorily the reaction cross-section for the interaction of a large number of nucleus pairs. The decomposition of the reaction cross-section into the contributions corresponding to subsequent impact parameter intervals suggests that at incident energies exceeding few tens of MeV/nucleon most of the reactions occur at an impact parameter substantially larger than the sum of the half-central-density radii of the interacting nuclei and involve a relatively small number of nucleons.

References

1. H.L. Bradt, B. Peters, Phys. Rev. **77**, 54 (1950).
2. P. Renberg *et al.*, Nucl. Phys. A **183**, 81 (1972).
3. R.M. DeVries, J.C. Peng, Phys. Rev. C **22**, 1055 (1980).
4. L.W. Townsend, J.W. Wilson, Radiat. Res. **106**, 283 (1986).
5. W.E. Frahn, Nucl. Phys. A **302**, 267 (1978).
6. R. Bass, *Nuclear Reactions with Heavy Ions* (Springer Verlag, Berlin, Heidelberg, New York, 1980).
7. W.W. Wilcke *et al.*, At. Data Nucl. Data Tables **25**, 389 (1980).
8. P.J. Karol, Phys. Rev. C **11**, 1203 (1975).
9. J.C. Peng, R.M. DeVries, N.J. DiGiacomo, Phys. Lett. B **98**, 244 (1981).

10. N.J. DiGiacomo, J.C. Peng, R.M. DeVries, Phys. Lett. B **101**, 383 (1981).
11. B.G. Harvey, Nucl. Phys. A **444**, 498 (1985).
12. A. Vitturi, F. Zardi, Phys. Rev. C **36**, 1404 (1987).
13. S.M. Lenzi, A. Vitturi, F. Zardi, Phys. Rev. C **40**, 2114 (1989).
14. S.K. Charagi, S.K. Gupta, Phys. Rev. C **41**, 1610 (1990).
15. S.K. Charagi, S.K. Gupta, Phys. Rev. C **46**, 1982 (1992).
16. S.K. Charagi, Phys. Rev. C **48**, 452 (1993).
17. Cai Xiangzhou *et al.*, Phys. Rev. C **58**, 572 (1998).
18. V.K. Lukyanov, B. Slowinski, E.V. Zemlyanaya, Phys. At. Nucl. **64**, 1273 (2001).
19. A.Sh. Ghazal, M.S M. Nour El-Din, M.Y.M. Hassan, Eur. Phys. J. A **19**, 221 (2004).
20. S. Fernbach, R. Serber, T.B. Taylor, Phys Rev. **75**, 1352 (1949).
21. R.J. Glauber, *Lectures on Theoretical Physics*, Vol. **I** (Interscience, New York, 1959).
22. J.L. Gammel, in *Fast Neutron Physics*, Part II, edited by J.B. Marion, J.L. Fowler (Interscience, New York, 1960) p. 2185.
23. M. Abramowitz, I.A. Segun, *Handbook of Mathematical Functions* (Dover, New York, 1968).
24. K. Kikuchi, M. Kawai, *Nuclear Matter and Nuclear Reactions*, (North-Holland, Amsterdam, 1968).
25. E. Gadioli, P.E. Hodgson, *Pre-equilibrium Nuclear Reactions* (Oxford Science Publications, Clarendon Press, Oxford, 1992).
26. J. Jaros *et al.*, Phys. Rev. C **18**, 2273 (1978).
27. W. True *et al.*, Phys. Rev. C **22**, 2462 (1980).
28. H.G. Bohlen *et al.*, Z. Phys. A **308**, 121 (1982).
29. M.E. Brandon *et al.*, J. Phys. G **9**, 197 (1983).
30. M. Buenerd *et al.*, Nucl. Phys. A **424**, 313 (1984).
31. S. Kox *et al.*, Phys. Rev. C **35**, 1678 (1987).
32. D.A. Roberts *et al.*, Phys. Rev. C **65**, 0446051 (2002).
33. C.C. Sahm *et al.*, Phys. Rev. C **34**, 2165 (1986).
34. A.J. Sjerk, Phys. Rev. C **33**, 2039 (1986).